Math 4200
Friday October 2

2.3 Homotopies, simply connected domains (rigorously); antiderivatives for analytic functions in simply-connected domains (rigorously); the Deformation Theorem (rigorously). We may not finish these notes today, but we will get close.

Announcements:

- (1) working on HW 4...
- (2) M, W 2.4 (not on exam 1; very important section)
- (3) Will go over an old exam W or Th next week.

On Wednesday we proved the

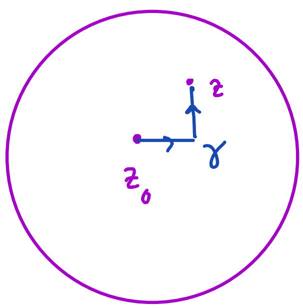
- <u>Rectangle Lemma</u> Let $f: D(z_0; r) \to \mathbb{C}$ be analytic. Let $R = [a, b] \times [c, d] \subseteq D(z_0, r)$ be a closed coordinate rectangle inside the disk. (i.e. $R = \{x + i\,y \mid a \le x \le b,\ c \le y \le d\} \subseteq D$.) Let $\gamma = \delta R$, oriented counterclockwise. Then

$$\int_\gamma f(z)\, dz = 0.$$

- We used Goursat's subdivision argument. If f had been C^1 we could've just used Green's Theorem.

Then we used the Rectangle Lemma to prove the

- <u>Local antiderivative Theorem</u> Let $f: D(z_0; r) \to \mathbb{C}$ be analytic. Then $\exists\ F : D(z_0; r) \to \mathbb{C}$ such that $F' = f$ in $D(z_0; r)$.



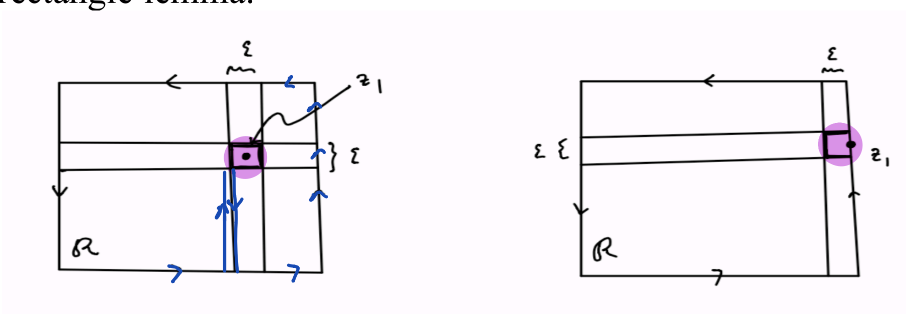
$$F(z) = \int_\gamma f(z)dz$$

For later (section 2.4): The Local antiderivative Theorem also holds if $f : D(z_0; r) \to \mathbb{C}$ is analytic except at a single point z_1 in the disk, where it is only known that f is continuous at z_1.

proof: The rectangle lemma + f continuous allows the construction of the antiderivative F. The rectangle lemma used the analyticity of f, but if there's just a single point z_1 where we don't have analyticity but do have that f is continuous (hence also bounded), we can still prove that the rectangle lemma holds for all rectangles. Here's how: Let R be chosen.

- If $z_1 \notin R$, there's no problem. (Goursat's argument only used subdivision within the rectangle.)

- If z_1 is in the interior of R or the boundary of R, subdivide and use a limiting argument with subrectangles and contour integral cancellations, and the boundedness of f near z_1 to deduce the rectangle lemma:



Let $\varepsilon > 0$, subdivide as indicated. Let R_{z_1} be the $\varepsilon \times \varepsilon$ rectangle as indicated above.

Apply the rectangle lemma on all other rectangles of the subdivision, note cancellation of contour integrals in the interior of R, and deduce

$$\oint_{\delta R} f(z)\, dz = \oint_{\delta R_{z_1}} f(z)\, dz.$$

And

$$\left| \int_{\delta R_{z_1}} f(z)\, dz \right| \le \int_{\delta R_{z_1}} |f(z)|\, |dz| \le M\, 4\, \varepsilon \to 0 \text{ as } \varepsilon \to 0\,,$$

where M is a local bound on $|f(z)|$ near z_1 which we have because f is continuous at z_1.

Today and Monday we define and discuss the notion of *homotopies;* use them to give a useable analysis definition of *simply connected domains;* and then use the local antiderivative theorem to prove the *global antiderivative theorem for simply connected domains*. This theorem will follow from a more general *deformation theorem* that we also prove, about when contour integrals for an analytic function remain the same, when an initial contour is deformed via homotopy into a final contour. The deformation theorem will complement what we already know from section ~~3~~2.2, about contour replacement.

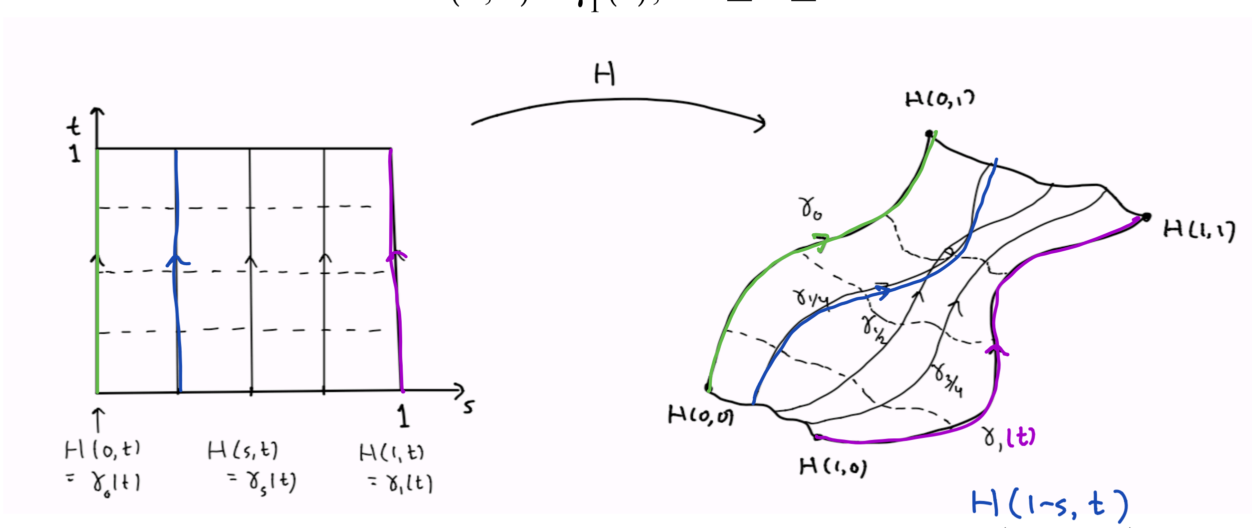
Two (continuous) contours are *homotopic* in a domain if one of them can be continuously deformed into the other one, within the domain. Precisely:

Def Let $A \subseteq \mathbb{C}$ be open and connected. Let $\gamma_0, \gamma_1 : [0, 1] \rightarrow A$ be continuous. paths.
Then γ_0 is homotopic to γ_1 in A if and only if

homotopy

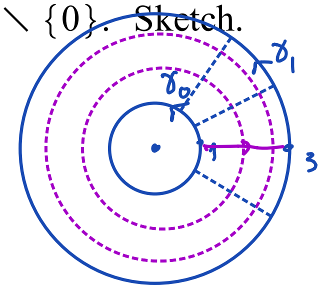
$$\exists\, H : \{(s, t) | \ 0 \leq s \leq 1,\ 0 \leq t \leq 1\} \rightarrow A \ \text{ continuous, such that}$$

- $H(0, t) = \gamma_0(t), \quad 0 \leq t \leq 1$
- $H(1, t) = \gamma_1(t), \quad 0 \leq t \leq 1$



Note: We call H the homotopy from γ_0 to γ_1. The composition $\cancel{H(1-t, s)}$ $H(1-s, t)$ is then a homotopy from γ_1 to γ_0. In the definition we use the unit square as the domain for the homotopy, but we could use any coordinate rectangle in the $s-t$ plane, because one can always rescale and translate.

Example: Find a homotopy between the unit circle and the circle of radius 3, in $\mathbb{C} \setminus \{0\}$. Sketch.



$0 \leq s \leq 1$

$H(s,t) = (2s+1)e^{it}$

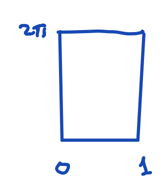
$0 \leq s \leq 1$

$\gamma_0(t) = e^{it}$

$\gamma_1(t) = 3e^{it}$

$0 \leq t \leq 2\pi$

$\gamma_s(t), \ 0 \leq t \leq 2\pi$



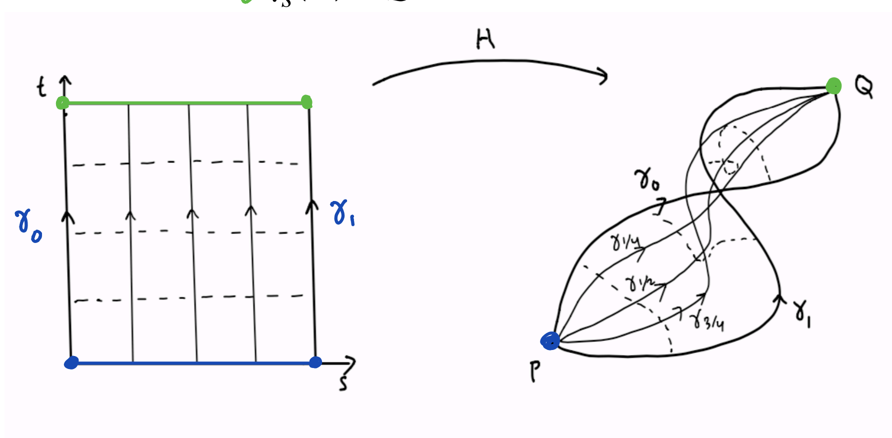
Special cases of homotopies:

Def The paths γ_0 and γ_1 are homotopic *with fixed endpoints* in A if and only if there are points $P, Q \in A$ with

- $\gamma_0(0) = \gamma_1(0) = P$
- $\gamma_0(1) = \gamma_1(1) = Q$

and $\exists$ homotopy $H(s, t) = \gamma_s(t)$ from the unit square to A such that

$$\begin{cases} \gamma_s(0) = P & \forall\ 0 \leq s \leq 1 \\ \gamma_s(1) = Q & \forall\ 0 \leq s \leq 1 \end{cases}$$

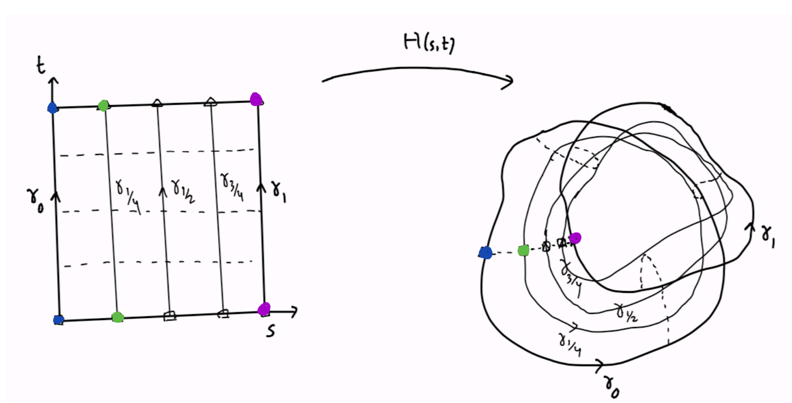


Def The paths γ_0 and γ_1 are homotopic *as closed curves* in A if and only if

- $\gamma_0(0) = \gamma_0(1)$ and $\gamma_1(0) = \gamma_1(1)$

and $\exists$ homotopy $H(s, t) = \gamma_s(t)$ from of closed curves from the unit square to A, i.e. such that

$$\gamma_s(0) = \gamma_s(1)\ \forall\ 0 \leq s \leq 1.$$



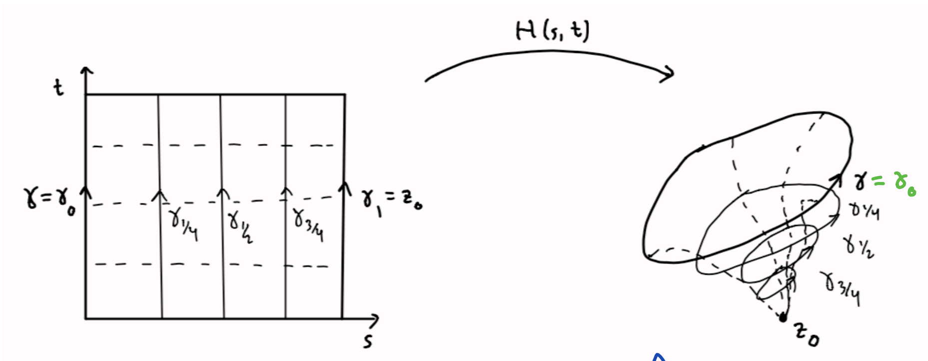
Def A connected open set A is *simply connected* if and only if every (cont.) closed curve $\gamma : [0,1] \to A$ is homotopic as a closed curve to some point $z_0 \in A$, i.e.

$\exists\ H : [0,1] \times [0,1] \to A$ continuous, such that

$$\gamma_0(t) = H(0,t) = \gamma(t), \qquad 0 \le t \le 1$$

$$\gamma_1(t) = H(1,t) = z_0, \qquad 0 \le t \le 1$$

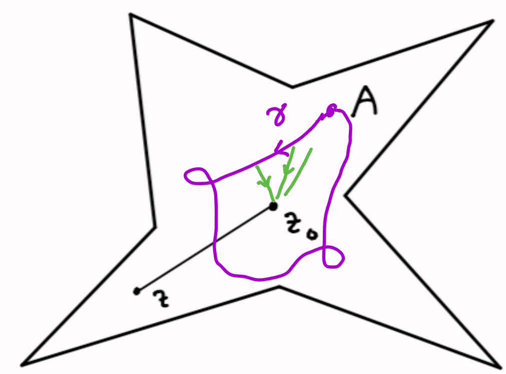
$$\gamma_s(0) = \gamma_s(1) \quad H(s,0) = H(s,1), \qquad 0 \le s \le 1\ .$$



Def A domain A is called *starshaped* with respect to $z_0 \in A$ if and only $\exists\ z_0 \in A$ such that $\forall\ z \in A$ the line segement $\{(1-s)z + s\,z_0 \mid 0 \le s \le 1\} \subseteq A$.

$s = 0 : z$

$s = 1 : z_0$



Example: Check that if A is starshaped, then A is simply connected.

$\gamma : [a,b] \to A$

$\gamma(t)$

$$H(s,t) = (1-s)\gamma(t) + s\,z_0$$

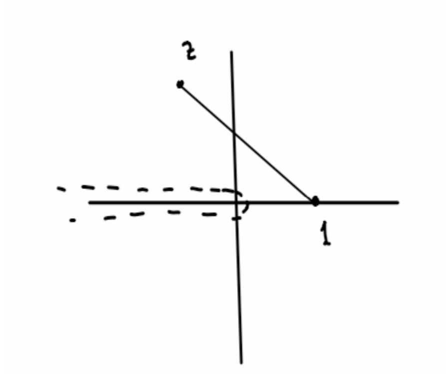
$$0 \le s \le 1$$

$$= \gamma(t) + s(z_0 - \gamma(t))$$

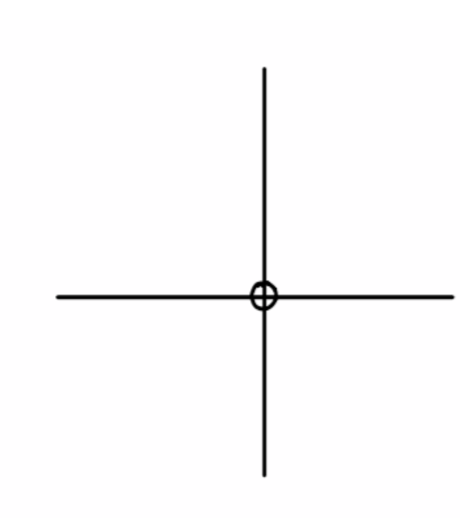
$H(0,t) = \gamma(t)$

$H(1,t) = z_0$

Some of our favorite branched domains are star-shaped, so also are simply connected. By the antiderivative theorem for simply connected domains - which we are about to prove rigorously as opposed to the section 2.2 arguments - that means if we have analytic functions in star-shaped branched domains, they will have antiderivatives.



2.3.1 *Example* $\mathbb{C} \setminus \{0\}$ is not simply-connected. This is a homework problem based on a proof by contradiction using the function $\dfrac{1}{z}$ and the antiderivative theorem for simply connected domains.

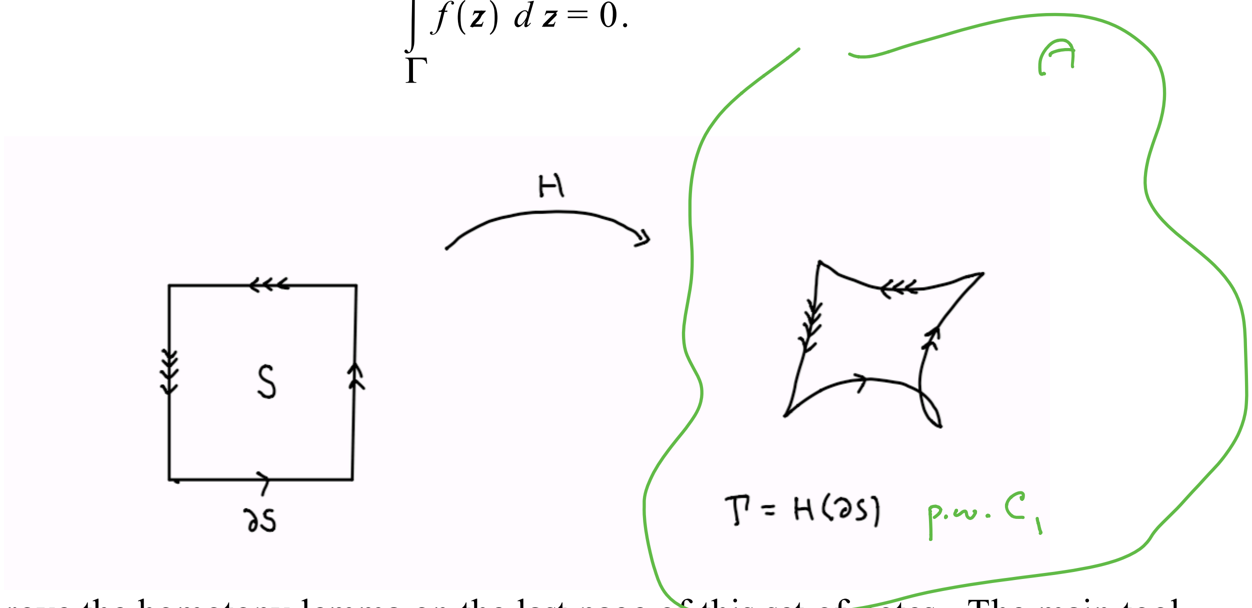


Homotopy Lemma Let $A \subseteq \mathbb{C}$ be open and connected. Let $f: A \to \mathbb{C}$ be analytic. Let

$$S = \{(s, t) \mid 0 \le s \le 1,\ 0 \le t \le 1\} \text{ and }$$
$$\delta S$$

denote the unit square and its boundary, oriented counterclockwise. Let $H: S \to A$ be continuous, with $\Gamma := H(\delta S)$ a piecewise C^1 contour. Then

$$\int_\Gamma f(z)\, dz = 0.$$

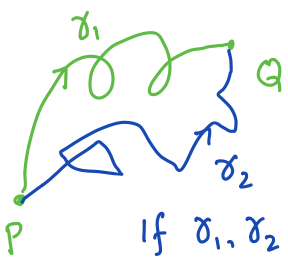


We will prove the homotopy lemma on the last page of this set of notes. The main tool is the local antiderivative theorem. The homotopy lemma is the key step for the main two theorems of section 2.3:

Theorem 1 Anti derivatives for analytic functions in simply connected domains: Let $A \subseteq \mathbb{C}$ be simply connected. Let $f: A \to \mathbb{C}$ analytic. Then $\exists\ F: A \to \mathbb{C}$ such that $F' = f$ in A.

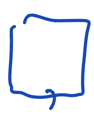
proof: It suffices to prove that contour integrals are path independent, or equivalently that whenever $\gamma: [a, b] \to A$ is a *closed* piecewise C^1 curve - which we can assume is actually parameterized on the interval $[0, 1]$ - then

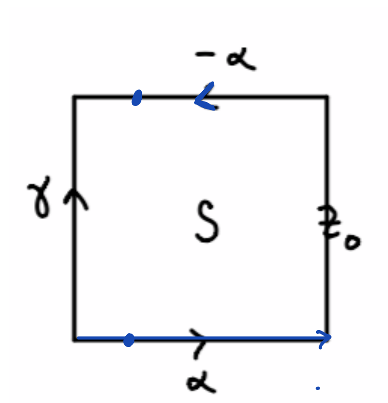
$$\int_{\gamma = \gamma_1 - \gamma_2} f(z)\, dz = 0. \implies \int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$$



If γ_1, γ_2 connect P to Q
Then $\gamma_1 - \gamma_2$ is closed p.w. C^1 contour.

By simple-connectivity, for such a γ there is a homotopy of γ to a fixed point $z_0 \in A$: We label the sides of the unit square by the images under this homotopy. Note that the closed curve condition means that if the lower directed segment is mapped to a curve α, then the upper directed curve is mapped to $-\alpha$.



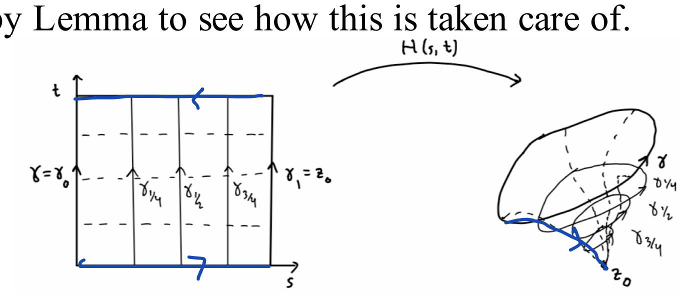


By the homotopy lemma

$$\underline{0 = \int_{\Gamma} f(z)\, dz} = \cancel{\int_{\alpha} f(z)\, dz} + \underbrace{\int_{z_0} f(z)\, dz}_{= 0} - \cancel{\int_{\alpha} f(z)\, dz} - \int_{\gamma} f(z)\, dz = -\int_{\gamma} f(z)\, dz .$$

Q.E.D.

Technical note: Since the homotopy H is only assumed to be continuous, the curves α, $-\alpha$ may not be piecewise C^1, so the contour integrals over them may not exist. See the proof of the Homotopy Lemma to see how this is taken care of.

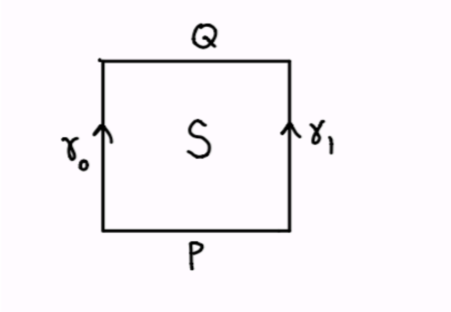


Theorem 2 Deformation Theorem Let $A \subseteq \mathbb{C}$ be open and connected (but not necessarily simply connected). Let $f: A \to \mathbb{C}$ analytic. If the two piecewise C^1 curves γ_0, γ_1 are homotopic in A, either with *fixed endpoints* or as *closed curves*, then

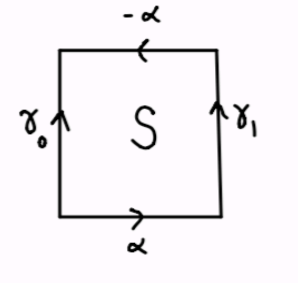
- $$\int_{\gamma_0} f(z)\, dz = \int_{\gamma_1} f(z)\, dz$$

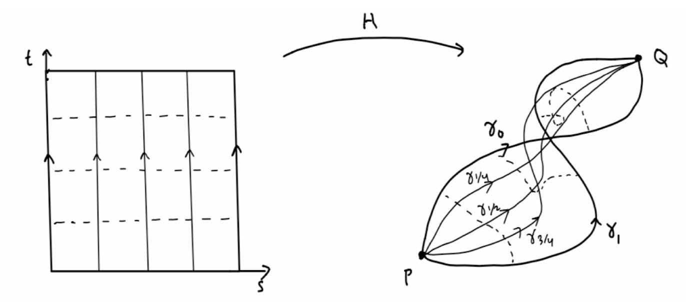
proof: Use the homotopy lemma on these two diagrams. Again, the edges of the unit square are labeled by their images under the homotopy:

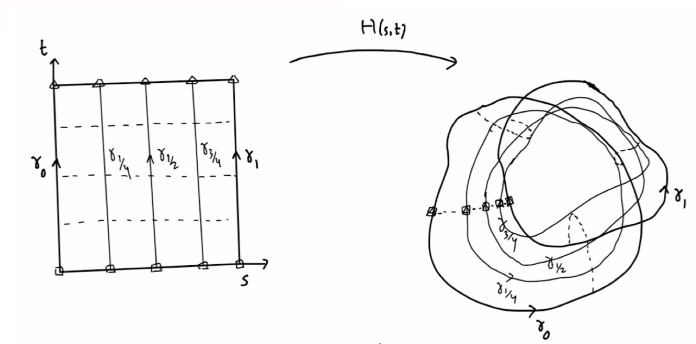
fixed end pt



as closed curves







Homotopy Lemma

$$\Rightarrow \cancel{\int_P f(z)dz} + \int_{\gamma_1} f(z)dz + \cancel{\int_Q f(z)dz} - \int_{\gamma_0} f(z)dz = 0$$

■

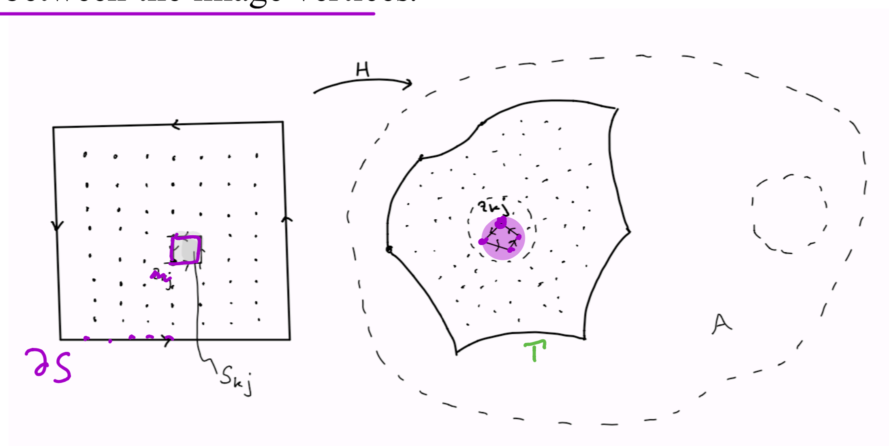
$$\cancel{\int_\alpha f(z)dz} + \int_{\gamma_1} f(z)dz + \cancel{\int_{-\alpha} f(z)dz} + \int_{-\gamma_0} f(z)dz = 0$$

■

proof of the homotopy lemma: Subdivide S into n^2 subsquares of side lengths n^{-1}. The dots in the diagram on the left indicate their vertices. number the squares as you would a matrix, and let S_{kj} be a typical subsquare, with z_{kj} be the image under the homotopy of its lower left corner. Since H is continuous and S is compact, the image $H(S) \subseteq A$ is compact. Write

- $H(\delta S) = \Gamma$
- $H(\delta S_{kj}) = \Gamma_{kj}$.

Replace any of the four subarcs of each Γ_{kj} which are not C^1 with constant speed line segment paths between the image vertices.



By interior cancellation,

$$\int_{\Gamma} f(z)\, dz = \sum_{k,j} \int_{\Gamma_{kj}} f(z)\, dz.$$

Note:

1) $H(S)$ is compact, $H(S) \subseteq A$ open, so by the Positive Distance Lemma you're proving in this week's homework

$$\exists\, \varepsilon > 0 \text{ such that } \forall\, z \in H(S),\ D(z; \varepsilon) \subseteq A.$$

2) H is continuous on S so H is uniformly continuous. Thus for ε as in (1),

$$\exists\, \delta > 0 \text{ such that } \|(s,t) - (\tilde{s}, \tilde{t})\| < \delta \Rightarrow |H(s,t) - H(\tilde{s}, \tilde{t})| < \varepsilon.$$

3) If n is large enough so that the diagonal length of the subsquares is less than δ, then each

$$H(S_{kj}) \subseteq D(z_{kj}; \varepsilon) \subseteq A,\ z_{kj} = H(s_k, t_j).$$

4) By the local antidifferentiation theorem in $D(z_{kj}; \varepsilon)$, each

$$\int_{\Gamma_{kj}} f(z)\, dz = 0 \Rightarrow \int_{\Gamma} f(z)\, dz = 0. \qquad \text{Q.E.D.!!!}$$

$$\int_{\Gamma_{kj}} f(z)\, dz = F_{kj}(P) - F_{kj}(P) = 0.$$